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Quantum Measurement I. The Measuring Process and the Interpretation of Quantum Mechanics

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Quantum mechanics and its interpretation are connected in a manifold way by the measuring process. The measuring apparatus serve as a means for the verification of the theory and are considered as physical objects also subject to the laws of this theory. On the basis of this interrelation some parts of the interpretation can be derived from other parts by means of quantum theory. On the other hand there are interpretations which must be excluded on the basis of the quantum theory of measurement.

1. INTRODUCTION

Quantum mechanics and its interpretation are connected in a manifold way by the measuring process. The interpretation provides a relation between the formalism of the object theory and experimental results. This relation is based on some particular features of quantum measurements and it shows in which way theoretical predictions can be verified by experimental means. On the other hand, the measuring apparatus are, considered as physical objects, subject to the laws of quantum theory, which leads to a quantum theory of quantum measurement. This theory has an important influence on the possibilities and limitations of various interpretations of the object theory. In fact it turns out that some parts of the interpretation are not independent but derivable from other parts by means of quantum object theory. On the other hand, there are interpretations which must be excluded on the basis of the quantum theory of measurement.

The measurement-induced interrelations between quantum object theory and its interpretation, i.e., its metatheory, express some self-referentiality of quantum mechanics. The interpretation of the theory is influenced in

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many ways by the properties of the measuring instruments which are, considered as physical objects, subject to the laws of quantum object theory (Dalla Chiara, 1977; Peres and Zurek, 1982). This twofold connection between the object theory and its metatheory can lead to a self-referential consistency of the theory, but also to self-referential inconsistencies. Both cases appear in quantum mechanics.

2. INTERPRETATIONS

2.1. Minimal Interpretation

The minimal interpretation I_M is the weakest possible interpretation of the quantum mechanical formalism and it is contained in any other consistent interpretation. It is in the spirit of the empiricism of David Hume and the positivism of Ernst Mach and was advocated in particular by Niels Bohr. It avoids any statements about the properties of individual objects and instead refers only to observational data, i.e., to measuring outcomes (Busch *et al.*, 1991).

Let S be an object system prepared in a pure state φ and $A = \sum a_i P[\varphi^{a_i}]$ a discrete nondegenerate observable. If φ is an eigenstate φ^{a_i} of A and the system S possesses the value a_i of A, then the *calibration postulate* (C) requires that a measurement of A at the system S with preparation $\varphi = \varphi^{a_i}$ leads to a pointer value Z_i of the measuring apparatus indicating that the system S had the eigenvalue a_i and the state φ^{a_i} before the measurement. If φ is not an eigenstate of A, $[P[\varphi], A]_- \neq 0$, then it is not possible to predict with certainty the pointer value after a measurement of A. In this case the *pointer objectification postulate* (PO) requires that after an A-measurement the pointer has some objective value Z_i, even if it cannot be predicted with certainty. Moreover, the *probability reproducibility postulate* (RP) requires that the probability distribution $p(\varphi, a_i) = \mathrm{tr}\{P[\varphi] \cdot P[\varphi^{a_i}]\}$ which is induced by the preparation φ and the observable A is reproduced in the statistics of the pointer values $Z_i^{(n)}$ after A-measurements on N equally prepared systems $S^{(n)}(\varphi)$ with $n = 1, 2, \ldots, N$.

2.2. Realistic Interpretation

In case of repeatable measurements of a discrete observable A the realistic interpretation I_R requires not only the postulates (C) and (PO), but also that the object system S possesses objectively an A-value a_i and that S is in the eigenstate φ^{a_i} after the A-measurement. This *system objectification postulate* (SO) leads to a description of the object system which is not necessarily contained in the minimal interpretation I_M.

According to the interpretation I_R, the repeatable measuring process of A transforms the initial state φ into a state φ^{a_i} which belongs to the A-value a_i. The quantum theory of measurement has to show under which conditions this additional postulate (SO) can be justified. Moreover, on the basis of the system objectification the probability reproducibility postulate (RP) is extended in the realistic interpretation such that the initial probability distribution $p(\varphi, a_i)$ is not only reproduced in the statistics of the pointer values, but also in the statistics of the A-values a_i of the object system $S^{(n)}$ after the measurement. The reproduction of the probability distribution in the statistics of system values $a_i^{(n)}$ is denoted here by (RS) (Busch *et al.*, 1991).

2.3. Objectifying Realistic Interpretation

In the realistic interpretation I_R nothing is said about the A-value of S before the measurement, i.e., in the state φ, except when φ is an eigenstate of A. In this situation one could tentatively assume that in addition to the interpretations I_M and I_R a certain value a_i of A pertains *objectively* to the system S in the state φ, but that this value is subjectively unknown to the observer, who knows only the probability $p(\varphi, a_i)$ of the value a_i. The probability distribution would then express the subjective ignorance of the observer about the objectively decided value of A. This attribution of a certain A-value to the system S will be called *weak objectification* (Busch and Mittelstaedt, 1991; Busch *et al.*, 1992).[2]

Under this hypothesis the system $S(\varphi)$ would possess the property a_i in a potential sense such that a measuring process would actualize the value a_i with the probability $p(\varphi, a_i)$. This is the content of the objectifying realistic interpretation I_{OR}. However, it turns out, that this extension of the realistic interpretation I_R is no longer consistent with quantum mechanics. This can be seen in the following way.

If the object system S with Hilbert space H_S is in a pure state $W = P[\varphi]$, $\varphi \in H_S$, and if φ is not an eigenstate of the observable $A = \sum a_i P[\varphi^{a_i}]$, then we assume that one of the values a_i *pertains objectively* to the system S, but that the observer knows only its probability $p(\varphi, a_i)$. From this assumption it follows for the probability of any other observable $B = \sum b_k P[\psi^{b_k}]$ with $[B, W] \neq 0$ and $[B, A] \neq 0$ that

$$p(\varphi, b_k) = \mathrm{tr}\{\textstyle\sum P[\varphi^{a_i}]P[\varphi]P[\varphi^{a_i}]P[\psi^{b_k}]\} \qquad (1)$$

[2]This terminology is used here, since one could also make the *strong objectification hypothesis* that the system not only possesses an eigenvalue of A, but that S is actually in an eigenstate of A. In the present context this hypothesis has, however, the same consequences as the weak objectification.

Therefore, weak objectification implies vanishing interference terms

$$p_{\text{int}}(\varphi; b_k, A) := p(\varphi, b_k) - \text{tr}\{\sum P[\varphi^{a_i}]P[\varphi]P[\varphi^{a_i}]P[\psi^{b_k}]\}$$

$$= \text{tr}\left\{\sum_{i \neq j} P[\varphi^{a_i}]P[\varphi]P[\varphi^{a_j}]P[\psi^{b_k}]\right\} = 0 \tag{2}$$

The assumption of weak objectification applied to A requires that equation (2) holds for arbitrary observables B. But this is known to be true only for eigenstates φ of A. In all other cases there are some observables B such that $p_{\text{int}}(\varphi; b_k, A) \neq 0$ and (2) is violated. Hence, for arbitrary φ the hypothesis of weak A-objectification is not consistent with quantum mechanics.

3. QUANTUM THEORY OF UNITARY MEASUREMENTS

Here we discuss a quantum mechanical model for the measuring process making use of unitary premeasurements (Beltrametti *et al.*, 1990; Busch *et al.*, 1991). Consequently, the object system S and the measuring apparatus M are considered as proper quantum systems with Hilbert spaces H_S and H_M, respectively. Let $\varphi \in H_S$ be the initial preparation of S, and $\Phi \in H_M$ the initial preparation (the neutral state) of M before the measurement. We then consider the measuring process of the ordinary, discrete nondegenerate observable $A = \sum a_i P[\varphi^{a_i}]$ and a pointer observable $Z = \sum Z_i P[\Phi_i]$ with nondegenerate spectrum. Using the complete orthonormal system $\{\varphi^{a_i}\}_i$ of states, one can expand the initial state of S as $\varphi = \sum c_i \varphi^{a_i}$ with $c_i = (\varphi^{a_i}, \varphi)$.

For the quantum mechanical description of the measuring process we distinguish several subsequent steps.

In step I, the *preparation*, the compound system $C = S + M$ is in the state $\Psi(S + M) = \varphi(S) \otimes \Phi(M)$ with $\Psi \in H_S \otimes H_M$.

In step II, the *premeasurement*, the system S and M are in interaction, which is described here by the unitary operator $U(t) = \exp(-i/\hbar H_{\text{int}} t)$ acting on the state $\Psi(S + M)$ within the time interval $0 \leq t \leq t'$. Hence we obtain $\Psi'(S + M) = U(t')(\varphi \otimes \Phi)$ for the compound state after the premeasurement. In order to further determine the state $\Psi'(S + M)$, we apply the *calibration postulate* (C). According to this postulate, for a system S with preparation $\varphi = \varphi^{a_i}$ in the interpretation I_M it holds that $U(\varphi^{a_i} \otimes \Phi) = \varphi'_i \otimes \Phi_i$, where the states $\varphi'_i \in H_S$ are not necessarily eigenstates of the observable A. However, for the interpretation I_R, which refers to repeatable measurements, one has $U(\varphi^{a_i} \otimes \Phi) = \varphi^{a_i} \otimes \Phi_i$. In the general case one thus obtains

$$\Psi' = U(\varphi \otimes \Phi) = \sum (\varphi^{a_i}, \varphi) U(\varphi^{a_i} \otimes \Phi) = \sum c_i \varphi'_i \otimes \Phi_i$$

with $\varphi'_i = \varphi^{a_i}$ for repeatable measurements. By specifying the object state φ'_i, one can characterize different kinds of premeasurements (Beltrametti *et al.*, 1990; Busch *et al.*, 1991). For the sake of simplicity we will deal here mainly with repeatable measurements, such that $\Psi'(S+M) = \sum c_i \varphi^{a_i} \otimes \Phi_i$. Finally, we note that for any observable A there exist a unitary operator U which provides an A-premeasurement of the discussed kind (Busch *et al.*, 1991).

In step III of the measuring process, *objectification and reading*, the two systems S and M are again dynamically independent but still correlated. As subsystems of the compound system $C = S + M$ in the state $\Psi'(S+M) = \sum c_i \varphi^{a_i} \otimes \Phi_i$, they can be described by the reduced (mixed) states

$$W'_S = \sum |c_i|^2 P[\varphi^{a_i}], \qquad W'_M = \sum |c_i|^2 P[\Phi_i]$$

respectively. According to the *pointer objectification postulate* (PO), after the measurement the pointer possesses some objective value Z_i indicating the measuring result a_i. This means that the mixed state W'_M must describe a *Gemenge* $\Gamma(W'_M)$, i.e., a mixture of states Φ_i such that actually one of the states Φ_i pertains to the apparatus M. Since we are dealing here with repeatable measurements, the strong correlations between S and M after the premeasurement imply the *system objectification postulate* (SO). This means that also the mixed state W'_S should describe a Gemenge $\Gamma(W'_S)$, i.e., a mixture of states φ^{a_i} such that the object system S is actually in one of the eigenstates φ^{a_i} of A and possesses the eigenvalue a_i.

4. PROBABILITY REPRODUCIBILITY

4.1. The Probability Interpretation

The calibration postulate $U(\varphi^{a_i} \otimes \Phi) = \varphi^{a_i} \otimes \Phi_i$ implies that the post-premeasurement state of the system $S+M$ reads $\Psi' = \sum (\varphi^{a_i}, \varphi)\varphi^{a_i} \otimes \Phi_i$. However, the meaning of the coefficients $c_i = (\varphi^{a_i}, \varphi)$ is still open. If $W = P[\varphi]$ is the preparation of S and if A is a discrete observable $A = \sum a_i P[\varphi^{a_i}]$ with $\{a_i\} = X^A$, then the real positive numbers $p(\varphi, a_i) = \operatorname{tr}\{P[\varphi]P[\varphi^{a_i}]\} = |(\varphi^{a_i}, \varphi)|^2$ are probabilities in the formal sense, i.e., the mapping

$$p^A_\varphi = B(X^A) \to [0, 1], \qquad a_i \to p(\varphi, a_i) = \operatorname{tr}\{P[\varphi]P[\varphi^{a_i}]\}$$

is a probability measure satisfying the Kolmogorov axioms [$B(X^A)$ is the Borel algebra of X^A]. The interpretation of the probability distribution $p(\varphi, a_i)$ can then be obtained from the interpretations I_{M} and I_{R}, in particular from the *probability reproducibility conditions* (RP) and (RS).

According to the condition (RP) the probability (φ, a_i) is reproduced in the p-statistics of the pointer values Z_i. In other words, the number $p(\varphi, a_i)$ is the probability to find after the measuring process the pointer value Z_i which indicates that the A-value a_i was measured. This means that if one would perform a series of N measurements of the observable A on equally prepared systems $S^{(n)}$ $(n = 1, 2, \ldots, N)$, the relative frequency $f^N(W, Z_i)$ of the pointer values Z_i would approach the probability $p(\varphi, a_i)$ for $N \to \infty$ (van Fraassen, 1979).

For repeatable measurements the condition (RP) can be extended to the stronger condition (RS). The probability $p(\varphi, a_i)$ is then not only reproduced in the statistics of the pointer values Z_i, but also in the statistics of the A-values a_i of the system S after the measuring process. The number $p(\varphi, a_i)$ is then also the probability to find after the measuring process the value a_i pertaining to S. This means that if one would perform a series of N measurements of A on equally prepared systems $S^{(n)}$, the relative frequency $f^N(W, a_n)$ of systems that after the measurement possess the value a_n would approach the value $p(\varphi, a_n)$ for $N \to \infty$ (van Fraassen, 1979).

4.2. The Metatheorem of the Minimal Interpretation

Within the interpretation I_M the calibration postulate (C) corresponds to a probability-free "weak minimal interpretation" I_M^0 which merely states that if a system is in an eigenstate φ^{a_i} of A, then the system possesses the A-value a_i. A measurement of A will then lead with certainty to the pointer value Z_i which indicates the measuring result a_i. For arbitrary preparations one has to add the postulates (PO) and (RP), which lead to the full minimal interpretation I_M containing also the probability interpretation. However, it can be shown that the postulate (RP) and the corresponding probability interpretation are no additional assumptions, but follow from I_M^0 and (PO) by means of quantum mechanics. This means that the quantum mechanical object theory and the probability-free weak minimal interpretation I_M^0 allow one to derive the full interpretation I_M which also refers to probabilities.

In order to deduce this "metatheorem" we presuppose that quantum mechanics is a complete and universally valid theory which can equally be applied to a single object system as well as to a many-body system $S^{(N)}$ consisting of N equally prepared system $S_i = S_i(W)$. The probability interpretation states that a sequence of N measurements of A on systems $S_i(W)$ would lead to a relative frequency $f^N(W, Z_i)$ of pointer values Z_i which approach the probability $p(\varphi, a_i)$ for $N \to \infty$. Hence one has to investigate whether in a compound system $S^{(N)}$ of systems $S_i(W)$ the relative frequency $f^N(W, Z_i)$ of measuring results a_i indicated by Z_i approaches the value $p(\varphi, a_i)$.

In order to study this problem we consider N independent systems S_i with Hilbert spaces H_{S_i} and the discrete observable $A = \sum a_i P[\varphi^{a_i}]$ with $\{a_i\}_i = X^A$. If the systems S_i are equally prepared in states $\varphi^{(i)} \in H_{S_i}$ the compound system $S^{(N)} = S_1 + S_2 + \cdots + S_N$ can be described by the state $(\varphi)^N = \varphi^{(1)} \otimes \varphi^{(2)} \otimes \cdots \otimes \varphi^{(N)}$. A premeasurement of A transforms the initial state $W_{S_i} = P[\varphi^{(i)}]$ of the system S_i into the mixed state $W'_{S_i} = \mathrm{tr}_M \{P[U(\varphi^{(i)} \otimes \Phi)]\}$ and the initial state Φ of M into the mixed state $W'_M = \mathrm{tr}_{S_i}\{P[U(\varphi^{(i)} \otimes \Phi)]\} = \sum p(\varphi, a_i) P[\Phi_i]$ with final states Φ_i corresponding to pointer values Z_i. This premeasurement can be extended to a premeasurement of the observable $A^{(N)} = A_1 \otimes A_2 \otimes \cdots \otimes A_N$ on the compound system $S^{(N)}$ in the state $(\varphi)^N$. The measuring outcome is then given by the sequence $\{Z_{l_1}, Z_{l_2}, \ldots, Z_{l_N}\}$ for the pointer values and the pointer states $\{\Phi_{l_1}, \Phi_{l_2}, \ldots, \Phi_{l_N}\}$ which indicate the measuring results a_{l_i}. By $l = \{l_1, l_2, \ldots, l_N\}$ we denote an index sequence such that $a_{l_i} \in X^A$. In the N-fold tensor product Hilbert space $H_M^{(N)}$ of the apparatus the states

$$\Phi_l^{(N)} = \Phi_{l_1} \otimes \Phi_{l_2} \otimes \cdots \otimes \Phi_{l_N} \in H_M^{(N)} \qquad \text{with} \quad l_i \in l$$

are a complete orthonormal basis, where the states Φ_{l_i} are eigenstates of the pointer observable, i.e., $Z\Phi_{l_i} = Z_i \Phi_{l_i}$. For any product state $\Phi_l^{(N)}$ which is determined by a sequence l the relative frequency of some pointer value Z_k is given by

$$f^N(k, l) = 1/N \sum_{i=1}^{N} \delta_{l_i, k}$$

Since the states $\Phi_l^{(N)}$ are orthogonal and complete, we can use these states for defining a "relative frequency operator" in $H_M^{(N)}$,

$$F_k^{(N)} = \sum_l f^N(k, l) P[\Phi_l^{(N)}]$$

where the sum runs over all sequences $l = \{l_1, l_2, \ldots, l_N\}$ with $l_i \colon a_{l_i} \in X^A$. The eigenvalue equation $F_k^{(N)} \Phi_l^{(N)} = f^N(k, l) \Phi_l^{(N)}$ then shows that the relative frequency of the pointer value Z_k is an objective property in the state $\Phi_l^{(N)}$ which is given by the eigenvalue $f^N(k, l)$ of $F_k^{(N)}$. This equation can be written equivalently as

$$\mathrm{tr}\{P[\Phi_l^{(N)}](F_k^{(N)} - f^N(k, l))^2\} = 0$$

In order to justify the minimal probability interpretation, which refers to a situation after the measuring process, we consider the reduced mixed states

$$W'_M = \sum p(\varphi, a_i) P[\Phi_i]$$

of the measuring apparatus. The sequence of N measurements of A is then described by the product state

$$(W'_M)^N = W'^{(1)}_M \otimes W'^{(2)}_M \otimes \cdots \otimes W'^{(N)}_M$$

where the upper index k of $W'^{(k)}_M$, say, refers to the measurement of system S_k. It is easy to see that $(W'_M)^N$ is in general not an eigenstate of $F^{(N)}_k$, which means that the relative frequency of the pointer value Z_k is not an objective property in the state $(W'_M)^N$. However, the following result holds:

Metatheorem I

$$\lim_{N\to\infty} \operatorname{tr}\{(W'_M)^N (F^{(N)}_k - p(\varphi, a_k))^2\} = 0$$

According to this theorem, in the limit of large N the postmeasurement product state $(W'_M)^N$ becomes an eigenstate of the relative frequency operator $F^{(N)}_k$ and the relative frequency of the pointer value Z_k approaches the probability $p(\varphi, a_k)$. Hence the probability distribution $p(\varphi, a_i)$ which is induced by the preparation φ and the observable A is reproduced in the statistics of the pointer values. In this sense Metatheorem I justifies the postulate (RP). For the proof of this metatheorem one must presuppose merely the weak minimal interpretation I^0_{M}, which corresponds to the calibration postulate (C) (Busch *et al.*, 1991; Mittelstaedt, 1991). Furthermore, for the interpretation of the metatheorem, the objectification of the pointer values, i.e., the postulate (PO), must be presupposed. Under these conditions Metatheorem I induces a self-referential consistency between object theory and metatheory.

4.3. The Metatheorem of the Realistic Interpretation

In the case of repeatable measurements one can apply the realistic interpretation I_{R}, according to which not only are the postulates (C), (PO), and (RP) fulfilled, but so are the stronger postulates (SO) and (RS). According to these postulates, the probability distribution $p(\varphi, a_i)$ is reproduced in the statistics of the system values a_i after the measurement. This means that a sequence of A-measurements on equally prepared systems $S_i(W)$ would lead to system values a_k and relative frequency $f^N(W, a_k)$ of which would approach the probability $p(\varphi, a_k)$ for $N \to \infty$. However, it can again be shown that the postulate (RS) and the corresponding realistic interpretation I_{R} are not additional assumptions, but follow (by means of quantum mechanics) from the weak realistic interpretation which makes use of the postulate (C).

In order to justify this metatheorem of the interpretation I_{R} we consider again N equally prepared systems $S_i(W)$, $W = P[\varphi]$ as a compound system $S^{(N)}$ in the state $(\varphi)^N \in H^{(N)}_M$. A unitary and repeatable premeasurement of A will then transform the initial states $W_{S_i} = P[\varphi^{(i)}]$ of S_i into the reduced mixed states

$$W'_{S_i} = \mathrm{tr}_M\{P[U(\varphi^{(i)} \otimes \Phi)]\} = \sum p(\varphi, a_i) P[\varphi^{a_i}]$$

with the final states φ^{a_i} and A-values a_i. A premeasurement of $A^{(N)}$ on the compound system $S^{(N)}$ in the state $(\varphi)^N$ will then lead to a sequence $\{a_{l_1}, a_{l_2}, \ldots, a_{l_N)}\}$ of outcomes and to a sequence of final states $\varphi_{l_i}^{(i)} \in H_{S_i}$ satisfying the eigenvalue equation $A\varphi_{l_i}^{(i)} = a_{l_i}\varphi_{l_i}^{(i)}$. By l_i we denote again the index values such that $a_{l_i} \in X^A$.

In the Hilbert space $H^{(N)}$ of the compound system $S^{(N)}$ the product states $\varphi_l^{(N)} = \varphi_{l_1}^{(1)} \otimes \varphi_{l_2}^{(2)} \otimes \cdots \otimes \varphi_{l_N}^{(N)} \in H_S^{(N)}$ with index sequences $l = \{l_1, l_2, \ldots, l_N\}$ form a complete and orthonormal basis. In a state $\varphi_l^{(N)}$ each subsystem S_i has a well-defined A-value a_{l_i}. The relative frequency of the eigenvalue a_k in the state $\varphi_l^{(N)}$ is then given by $f^N(k, l) = 1/N \sum \delta_{l_i,k}$. A relative frequency operator for the value a_k can be defined using the states $\varphi_l^{(N)}$ in $H_S^{(N)}$ by $f_k^{(N)} = \sum f^N(k, l) P[\varphi_l^{(N)}]$. In accordance with the calibration postulate, the eigenvalue equation $f_k^{(N)}\varphi_l^{(N)} = f^{(N)}(k, l)\varphi_l^{(N)}$ shows, that the relative frequency of the value a_k is an objective property of $S^{(N)}$ in the state $\varphi_l^{(N)}$. The eigenvalue equation can also be written in the equivalent form

$$\mathrm{tr}\{P[\varphi_l^{(N)}](f_k^{(N)} - f^N(k, l))^2\} = 0$$

For a justification of the interpretation I_{R} one has to consider the postpremeasurement state of the compound system $S^{(N)}$, which is given by the tensor product $(W'_S)^N = \bigotimes_{i=1} W'_{S_i}$ of mixed reduced states W'_{S_i}. In general the state $(W'_S)^N$ is not an eigenstate of $f_k^{(N)}$, which means that in the state $(W'_S)^N$ the relative frequency of a_k is not an objective property of the system $S^{(N)}$. However, for large values of N the postpremeasurement product state $(W'_S)^N$ becomes an eigenstate of $f_k^{(N)}$ and we have the following result.

Metatheorem II

$$\lim_{N \to \infty} \mathrm{tr}\{(W'_S)^N (f_k^{(N)} - p(\varphi, a_k))^2\} = 0$$

This theorem shows that for the limit $N \to \infty$ the system $S^{(N)}$ in the state $(W'_S)^N$ possesses an objective value of the relative frequency of a_k and that this value agrees with the probability $p(\varphi, a_k)$ which is induced by φ and A. The proof of this theorem is rather analogous to the proof of Metatheorem I and can be found in Busch *et al.* (1991) and Mittelstaedt (1991). It is based on the weak realistic interpretation I_{R}^0, which corresponds to the calibration postulate (C) and on the assumption that the mixed state W'_{S_i} can be understood as a description of a mixture of states

φ^{a_k}, an assumption which corresponds to the system objectification (SO). Under these conditions it follows that the probability distribution $p(\varphi, a_i)$ is reproduced in the statistics of the system values a_i after the A-measurement [(RS) postulate]. In this way a further self-referential consistency between the object quantum theory and its metatheory is induced by Metatheorem II.

5. THE NONOBJECTIFICATION THEOREMS

5.1. Nonobjectification in Pure States

The impossibility of a weak or strong objectification in pure states is important for any interpretation of quantum mechanics. According to the results of Section 2.3, a quantum system S in a pure state φ, which is not an eigenstate of the observable A, does not admit a weak or strong objectification of A. This means that the state $P[\varphi]$ must not be considered as an incomplete description of S, such that $S(W)$ is actually in an eigenstate φ^{a_i} of A or possesses a value a_i which is, however, not known to the observer.

On the basis of this interpretation the following problem arises. Even if the value of A and the corresponding eigenstate are not objective in the state W, the observable A can be measured by means of a convenient apparatus M. After the measuring process the system should be in an eigenstate of A, i.e., the measuring process must provide in some way the objectification of A. This is the content of the postulate (SO) in the interpretation I_{R}. Hence one should expect that the mixed state W'_S of S after the premeasurement admits a strong or weak ignorance interpretation. The system would then be in an eigenstate φ^{a_i} of A or it would at least possess one of the values a_i. In case of nonrepeatable measurements one would at least expect that the mixed state W'_M of the measuring apparatus admits a strong or weak ignorance interpretation, which means that M is in an eigenstate Φ_i of Z or that a value Z_i of Z pertains to the measuring apparatus. This corresponds to the postulate (PO) in the interpretation I_{M}.

5.2. Nonobjectification in Mixed States

In order to discuss the question of whether the mixed states W'_S and W'_M admit an ignorance interpretation, we consider the compound system $C = S + M$ with the initial preparation $\Psi(S+M) = \varphi \otimes \Phi$. After a unitary repeatable premeasurement the system $S + M$ is in the pure state

$$\Psi'(S+M) = U(\varphi \otimes \Phi) = \sum_i (\varphi^{a_i}, \varphi)\varphi^{a_i} \otimes \Phi_i$$

where U is the unitary operator which provides the premeasurement of A. Here we have assumed unitary and *repeatable* premeasurements. For unitary but *nonrepeatable* premeasurements in the compound state $\Psi'(S+M)$ one has to replace the states φ^{a_i} by more general states $\varphi'_i \in H_S$ which need not be eigenstates of A.

If the compound system $S+M$ is in the pure state $W' = P[\Psi'(S+M)]$, then the subsystems S and M are in the correlated reduced mixed states

$$W'_S = \text{tr}_M\{W'\} = \sum p(\varphi, a_i)P[\varphi^{a_i}], \qquad W'_M = \text{tr}_S\{W'\} = \sum p(\varphi, a_i)P[\Phi_i]$$

where $p(\varphi, a_i) = |(\varphi^{a_i}, \varphi)|^2$ are the probabilities for the values a_i and Z_i, respectively. One can then argue in the following way. The strong objectification hypothesis for the system S in the state W'_S with respect to the eigenstates φ^{a_i} of A implies that the compound system $S+M$ is in the mixed state

$$W_{\Psi'} = \sum p(\varphi, a_i)P[\varphi^{a_i} \otimes \Phi_i]$$

and not, as previously assumed, in the pure state Ψ'. Consequently, for an observable $B^* = \sum B_k^* P[\Psi_k]$ with $\Psi_k \in H_S \otimes H_M$ of the compound system, one would obtain for the probability $p(\Psi', B_k)$ of the value B_k

$$p(\Psi', B_k^*) = \text{tr}\{W_{\Psi'} P[\Psi_k]\} \tag{3}$$

However, this equation implies vanishing interference terms

$$p_{\text{int}}(\Psi', A^*, B_k^*) = \text{tr}\left\{\sum_{i \neq j} P[\varphi^{a_i} \otimes \Phi_i]P[\Psi']P[\varphi^{a_j} \otimes \Phi_j]P[\Psi_k]\right\} = 0$$

for arbitrary B^*. Since this is known to be true only for eigenstates Ψ' of the operator $A^* = A \otimes \mathbf{1}_M$, it follows that for arbitrary states Ψ' the strong ignorance interpretation applied to the mixed states W'_M and W'_S is inconsistent with quantum mechanics.

This result can be extended to the weak ignorance interpretation, i.e., to the attribution of a value a_i to the system S in the state W'_S. The reason is that the weak ignorance hypothesis applied to the state W'_S leads to the same equation (3) for the probability of a value B_k^* of the system $S+M$ (Busch *et al.*, 1992). Since this equation is in general inconsistent with quantum mechanics, the weak ignorance interpretation of the system S in the state W'_S is not tenable. This is the content of the following nonobjectification (NO) theorem:

(NO) Theorem I. Let S be an object system with state φ and $A = \sum a_i P[\varphi^{a_i}]$ a discrete observable such that φ is not an eigenstate of A. After a unitary and repeatable premeasurement, S is described by the state

$W'_S = \sum p(\varphi, a_i)P[\varphi^{a_i}]$, which does not admit a strong or weak ignorance interpretation.

An analogous result holds for the measuring apparatus in the reduced mixed state W'_M after the premeasurement. This result is of particular interest for unitary nonrepeatable premeasurements, since in this case only the pointer will be in the mixed state $W'_M = \sum p(\varphi, a_i)P[\Phi_i]$ with eigenstates Φ_i of the pointer observable, whereas the state W'_S of S does not correspond to a decomposition of eigenstates φ^{a_i} of A. Applied to the measuring apparatus M, it follows from the arguments mentioned above that it is not possible to assume that after the premeasurement of A the apparatus M is merely described by the mixed state W'_M but that M is objectively in one of the eigenstates Φ_i of Z. Moreover, it is impossible to assume that the system M in the state W'_M possesses objectively some pointer value Z_i which is merely subjectively unknown to the observer. We thus arrive at the following theorem.

(NO) Theorem II. After a unitary premeasurement of the observable A on the system S in the state φ the measuring apparatus M is in a mixed state $W'_M = \sum p(\varphi, a_i)P[\Phi_i]$, which does not admit a strong or weak ignorance interpretation.

The nonobjectification theorems are of particular importance for the interpretations I_R and I_M. NO Theorem I shows that by means of unitary and repeatable premeasurements the system objectification postulate (SO) cannot be fulfilled, i.e., unitarity (U) implies $\neg$(SO). Moreover, unitary premeasurements are also inconsistent with the pointer objectification postulate (PO), i.e., (U) implies also $\neg$(PO). It is obvious that these results represent a serious self-referential inconsistency of quantum mechanics, since by means of the object theory some parts of the metatheoretical interpretations I_R and I_M can be disproved.

6. CONCLUDING REMARKS

6.1. Probability Metatheorems

Quantum mechanics and the theory of unitary premeasurements show a remarkable self-referential consistency. Quantum object theory is not only in accordance with the statistical components of the interpretations I_M and I_R. Moreover, quantum mechanics allows one to derive the reproduction of the initial probability in the statistics of the measuring outcomes if merely the probability-free parts of the respective interpretations are presupposed. In particular, on the basis of quantum object theory the postulates (C) and (PO) imply (RP) corresponding to the minimal interpretation

I_M (Metatheorem I)—and the postulates (C) and (SO) imply (RS), which corresponds to the realistic interpretation (Metatheorem II).

6.2. Nonobjectification Theorems

On the other hand, quantum theory and the theory of unitary premeasurements allow one to disprove the objectification requirements of the interpretations I_R and I_M. In fact, in case of the realistic interpretation the unitarity (U) implies $\neg$(SO) (NO Theorem I) and in case of the minimal interpretation the unitarity (U) implies $\neg$(PO) (NO Theorem II). Since the objectification requirements (SO) and (PO) are essential parts of the interpretations I_R and I_M it follows that quantum object theory disproves its own interpretation. These contradictions between object theory and its metatheory indicate a serious self-referential inconsistency of quantum mechanics. It should be added that the conclusions $\neg$(SO) and $\neg$(PO) of the NO Theorems I and II also invalidate the importance of the probability Metatheorems I and II, since for the interpretation of these theorems the objectification requirements (SO) and (PO) must be fulfilled.

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